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Permutations as Minimal Powers of a Single-cycle Class-sum

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The structure-constant $[(p)]_n \cdot [(1)^{l_1}(2)^{l_2} \cdots (n)^{l_n}]_n|_{[(1)^{l_1-k_1}(2)^{l_2-k_2} \cdots (u)^{l_u+1} \cdots (n)^{l_n}]_n}$ that corresponds to the merging of the p cycles $(1)^{k_1}(2)^{k_2} \cdots (u-1)^{k_{u-1}}$, $(\sum_{i=1}^{u-1} k_i = p)$ into a single cycle of length $u = \sum_{i=1}^{u-1} ik_i$ is derived. The result is used to evaluate the structure constant $[(p)]_n^k|_{[(1)^{l_1}(p)^{l_2} \cdots ((i-1)(p-1)+1)^{l_i} \cdots]_n}$, where k is the minimal power of $[(p)]_n$ generating the class-sum $[(1)^{l_1}(p)^{l_2} \cdots ((i-1)(p-1)+1)^{l_i} \cdots]_n$ (i.e. $\sum_{i=1}^{n} (i-1)l_i = k$).

1. Introduction

The class-algebra of the symmetric group has attracted considerable attention because of its representation-theoretic applications, as well as those in combinatorics, graph theory, the quantum-mechanical many-body problem and in other areas. The product of a pair of class-sums C_1 and C_2 is a linear combination with non-negative integral coefficients $C_1 \cdot C_2|_{C_i}$ of class-sums C_i, which we write in the form

$$C_1 \cdot C_2 = \sum_{C_i \in CS_n} C_1 \cdot C_2|_{C_i} C_i$$

These coefficients are sometimes referred to as the structure constants of the class-algebra. Several attempts have been reported to derive these coefficients using their combinatorial significance [2–6, 11]. Closely related work involving the formulation of covering theorems, which specify the non-vanishing structure constants without explicitly evaluating them, was presented by Arad and Herzog [1].

A combinatorial study of the product of the class of transpositions with an arbitrary class of the symmetric group [10], followed by a similar study of the class of three-cycles [8], motivated the formulation of a conjecture concerning the form of the product of a single-cycle class-sum of arbitrary length with an arbitrary class-sum in the symmetric group algebra [9].

In the present paper, the structure-constant

$$[(p)]_n \cdot [(1)^{l_1}(2)^{l_2} \cdots (n)^{l_n}]_n|_{[(1)^{l_1-k_1}(2)^{l_2-k_2} \cdots (u)^{l_u+1} \cdots (n)^{l_n}]_n}$$

that corresponds to the merging of the p cycles $(1)^{k_1}(2)^{k_2} \cdots (u-1)^{k_{u-1}}$, $(\sum_{i=1}^{u-1} k_i = p)$ into a single cycle of length $u = \sum_{i=1}^{u-1} ik_i$ is derived. The result is used to evaluate the structure constant $[(p)]_n^k|_{[(1)^{l_1}(p)^{l_2} \cdots ((i-1)(p-1)+1)^{l_i} \cdots]_n}$, where k is the minimal power of $[(p)]_n$ generating the class-sum $[(1)^{l_1}(p)^{l_2} \cdots ((i-1)(p-1)+1)^{l_i} \cdots]_n$ (i.e. $\sum_{i=1}^{n} (i-1)l_i = k$). This is a generalization of the case $p = 2$ that was studied by Dénes [6], the graph-theoretical ramifications of which were recently elaborated upon by Moszkowski [12] and by Goulden and Pepper [7].

2. On Products with a Minimal Number of Cycles of a Single-cycle with an Arbitrary Class-sum

When a permutation of type $\pi^* \equiv (1)^{l_1}(2)^{l_2} \cdots (n)^{l_n}$ is multiplied by a p-cycle which possesses one index in common with each one of p cycles in π^*, it is found that these p cycles merge into one, i.e.,

$$(i_1, i_2, \ldots, i_p) \cdot (\underbrace{i_1, \ldots}_{\lambda_1})(\underbrace{i_2, \ldots}_{\lambda_2}) \cdots (\underbrace{i_p, \ldots}_{\lambda_p}) *** = (\overbrace{\underbrace{i_1, \ldots}_{\lambda_1}, \underbrace{i_2, \ldots}_{\lambda_2}, \ldots, \underbrace{i_p, \ldots}_{\lambda_p}}^{u}) ***$$

where $***$ denotes the cycles that have no index in common with the p-cycle $(i_1, i_2, \ldots, i_p)$ and the symbol

$$\underbrace{i_j, \ldots}_{\lambda_j}$$

denotes a specific sequence of λ_j indices, beginning with i_j. $\Lambda = \{\lambda_i; i = 1, 2, \ldots, p\}$ is the p-tuple of lengths of the p cycles that are being merged. The length of the cycle generated is $u = \sum_{i=1}^{p} \lambda_i = \sum_{i=1}^{u-1} i \cdot k_i$, where k_i is the number of cycles of length i in the set Λ. $\amalg$ denotes the particular sequence of u integers

$$\underbrace{i_1, \ldots}_{\lambda_1}, \underbrace{i_2, \ldots}_{\lambda_2}, \ldots, \underbrace{i_p, \ldots}_{\lambda_p}.$$

The resulting permutation is of type

$$\pi \equiv (1)^{l_1-k_1}(2)^{l_2-k_2} \cdots (u-1)^{l_{u-1}-k_{u-1}}(u)^{l_u+1} \cdots (n)^{l_n}.$$

The class-sum of permutations of type π will be denoted by $[\pi]_n$. We now prove the following.

THEOREM 1.

$$[(p)]_n \cdot [(1)^{l_1}(2)^{l_2} \cdots (n)^{l_n}]_n \big|_{[(1)^{l_1-k_1}(2)^{l_2-k_2} \cdots (u)^{l_u+1} \cdots (n)^{l_n}]_n} = u \frac{(p-1)!}{\prod_{i=1}^{u-1} k_i!} (l_u + 1)$$

where $\sum_{i=1}^{n} il_i = n$, $\sum_{i=1}^{u-1} ik_i = u$ *and* $\sum_{i=1}^{u-1} k_i = p$.

PROOF. In order to count the number of ways in which the particular cycle $(\amalg)$ could be obtained as a product of some p-cycle with an appropriate permutation of type π, we note that i_1 could be selected within $\amalg$ in u different ways. Independently, we can arrange the cycle lengths $\lambda_1, \lambda_2, \ldots, \lambda_p$ in $p!/\prod_{i=1}^{u-1} k_i!$ distinct ways. Having selected the initial index i_1 and a particular ordering of the cycle lengths we obtain a well-defined set of cycles by appropriately splitting $(\amalg)$, as well as a set of indices comprising the p-cycle $(i_1, i_2, \ldots, i_p)$. This p-cycle arises in p equivalent forms, related by cyclic permutations, by making all the different choices for i_1. To eliminate this multiple counting we divide by p.

Finally, assuming that the original permutation had l_u u-cycles, each one of the $l_u + 1$ u-cycles in the permutation π can be obtained from appropriate Λ-sets of cycles multiplied by appropriate p-cycles. Since these $l_u + 1$ u-cycles consist of disjoint sets of indices, they give rise to $l_u + 1$ distinct sets of products of p-cycles by permutations of type π^*.

Multiplying the factors u, $p!/\prod_{i=1}^{u-1} k_i!$, $1/p$ and $l_u + 1$ specified above, the theorem follows. □

We note in passing that this theorem is an immediate consequence of the conjecture presented in [9] concerning products of a single-cycle by arbitrary class-sums in the algebra of the symmetric group.

3. COEFFICIENTS OF CLASS-SUMS WITH A MINIMAL NUMBER OF CYCLES, IN THE EXPANSION OF $[(p)]_n^k$

In this section we evaluate the structure constants

$$[(p)]_n^k \big|_{[(1)^{l_1}(2)^{l_2} \cdots (n)^{l_n}]_n},$$

where $[(1)^{l_1}(2)^{l_2}\cdots(n)^{l_n}]_n$ is a class with the smallest possible number of cycles within the sets of classes in the expansion of $[(p)]_n^k$. Since the number of cycles can be reduced by $p-1$ upon each application of $[(p)]_n$ (i.e. by merging p cycles into one), the minimal number of cycles in $[(p)]_n^k$ is obtained if such merging takes place k times, starting from the identity, $[(1)^n]_n$, which consists of n cycles. Thus, the minimal number of cycles in class-sums appearing in the expansion of $[(p)]_n^k$ is $n-(p-1)k$.

Repeated fusions of p cycles at a time, starting from $[(p)]_n$, can only give rise to cycles the lengths of which can be written in the form $i(p-1)+1$, where i is a non-negative integer. The class-sums appearing after k such fusions, i.e. in $[(p)]_n^k$, will be specified in terms of the indices $l_1, l_2, \ldots, l_{i_{\max}}$, where l_i is the number of cycles of length $(i-1)(p-1)+1$, and

$$i_{\max}=1+\min\left(\left\lfloor\frac{n-1}{p-1}\right\rfloor, k\right).$$

The total number of cycles satisfies $\sum_{i=1}^{i_{\max}} l_i = n-k(p-1)$, which, noting that $\sum_{i=1}^{i_{\max}}((i-1)(p-1)+1)l_i = n$, can be written in the more convenient form

$$\sum_{i=1}^{i_{\max}}(i-1)l_i = k. \tag{1}$$

We shall now prove the following.

THEOREM 2.

$$[(p)]_n^k\big|_{[\prod_{i=1}^{i_{\max}}((i-1)(p-1)+1)^{l_i}]_n} = k!\prod_{j=2}^{i_{\max}}\left(\frac{((j-1)(p-1)+1)^{j-2}}{(j-1)!}\right)^{l_j}$$

where

$$i_{\max}=1+\min\left(\left\lfloor\frac{n-1}{p-1}\right\rfloor, k\right)$$

and the set of integers $\{l_2, l_3, \ldots, l_{i_{\max}}\}$ *satisfies* $\sum_{i=2}^{i_{\max}}(i-1)l_i = k$.

PROOF. Since the theorem holds for $k=1$ we assume that it holds for $k-1$ and prove that it holds for k. We start by writing the relation

$$\begin{aligned}
&[(p)]_n^k\big|_{[\prod_{j=1}^{i_{\max}}((j-1)(p-1)+1)^{l_j}]_n}\\
&=\sum_{i=2}^{i_{\max}}\sum_{q=1}^{p}\sum_{i_1<i_2<\cdots<i_q}\ \sum_{\substack{(\lambda_{i_1},\lambda_{i_2},\ldots,\lambda_{i_q})\vdash p\\ (\sum_{j=1}^{q} i_j\lambda_{i_j}=i+p-2)}}\\
&\quad\times[(p)]_n\cdot\left[\prod_{j=1}^{i_{\max}}((j-1)(p-1)+1)^{l_j+\lambda_j-\delta_{j,i}}\right]_n\bigg|_{[\prod_{j=1}^{i_{\max}}((j-1)(p-1)+1)^{l_j}]_n}\\
&\quad\times[(p)]_n^{k-1}\big|_{[\prod_{j=1}^{i_{\max}}((j-1)(p-1)+1)^{l_j+\lambda_j-\delta_{j,i}}]_n},
\end{aligned} \tag{2}$$

where $(\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_q})\vdash p$ means that $\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_q}$ form a partition of p, i.e. $\lambda_{i_1}\leqslant\lambda_{i_2}\leqslant\cdots\leqslant\lambda_{i_q}$ and $\lambda_{i_1}+\lambda_{i_2}+\cdots+\lambda_{i_q}=p$. Equation (2) evaluates the coefficient of $[\prod_{j=1}^{i_{\max}}((j-1)(p-1)+1)^{l_j}]_n$ in $[(p)]_n^k$ by summing over all the terms in $[(p)]_n^{k-1}$ that can give rise to that term. The p cycles which the last application of $[(p)]_n$ fuses are grouped into q subsets, with lengths specified by the indices $i_1, i_2, \ldots, i_q$, such that the number of cycles of length $(i_j-1)(p-1)+1$ is λ_{i_j}. i is the index specifying the length, $(i-1)(p-1)+1$, of the cycle inserted as a consequence of the fusion, and the condition $\sum_{j=1}^{q} i_j\lambda_{i_j} = i + p - 2$ specifies the fact that the length of that cycle is the sum of the lengths of the p cycles the fusion of which gave rise to it.

To evaluate the first factor on the right-hand side we use Theorem 1. For the second factor we use the induction hypothesis. We consequently obtain

$$
\begin{aligned}
&[(p)]_n^k\big|_{[\prod_{j=1}^{i_{\max}}((j-1)(p-1)+1)^{l_j}]_n} \\
&= k!\prod_{j=2}^{i_{\max}}\left(\frac{((j-1)(p-1)+1)^{j-2}}{(j-1)!}\right)^{l_j} \\
&\times\left\{\sum_{i=2}^{i_{\max}}\sum_{q=1}^{p}\sum_{i_1<i_2<\cdots<i_q}\ \sum_{\substack{(\lambda_{i_1},\lambda_{i_2},\ldots,\lambda_{i_q})\vdash p\\ (\sum_{j=1}^{q} i_j\lambda_{i_j}=i+p-2)}}\frac{(p-1)!}{k\prod_{j=1}^{q}\lambda_{i_j}}((i-1)(p-1)+1)l_i\right.\\
&\left.\times\prod_{j=2}^{i_{\max}}\left(\frac{((j-1)(p-1)+1)^{j-2}}{(j-1)!}\right)^{\lambda_j}\left(\frac{((i-1)(p-1)+1)^{i-2}}{(i-1)!}\right)^{-1}\right\}.
\end{aligned}
$$

We shall now show that the expression in curly brackets is equal to unity. This expression can be written in the form

$$
\frac{1}{kp}\sum_{i=2}^{i_{\max}}\frac{(i-1)!l_i}{((i-1)(p-1)+1)^{i-3}}\sum_{\substack{i_1,i_2,\ldots,i_p\\ (\sum_{j=1}^{p} i_j=i+p-2)}}\prod_{j=1}^{p}\left(\frac{((i_j-1)(p-1)+1)^{i_j-2}}{(i_j-1)!}\right),
$$

where the set $i_1, i_2, \ldots, i_p$ allows repetitions, so that the sum over this set of indices is equivalent to the sum over the original set $i_1, i_2, \ldots, i_q$ as well as over $\lambda_{i_1}, \lambda_{i_2}, \ldots, \lambda_{i_q}$, multiplied by $\prod_{j=1}^{q}\lambda_{i_j}!/p!$. Using the multinomial Abel identity ([13, p. 24])

$$
\sum_{\substack{i_1,i_2,\ldots,i_p=0\\ (\sum_{j=1}^{p} i_j=i)}}^{i}\prod_{j=1}^{p}\left(\frac{(i_j(p-1)+1)^{i_j-1}}{i_j!}\right)=\frac{p(i(p-1)+p)^{i-1}}{i!},
$$

we write the above expression in the form $(1/k)\sum_{i=2}^{i_{\max}}(i-1)l_i$, which, by equation (1), is equal to unity. □

Theorem 2 constitutes a generalization of Dénes' result [6], to which it reduces for $p = 2$. The original proof, due to Dénes, is based on establishing a correspondence with a certain enumeration problem involving edges of graphs. It may be instructive to consider the graph-theoretical implications of the present generalization, possibly along the lines discussed by Dénes [6] and, recently, by Moszkowski [12] and by Goulden and Pepper [7], but this is beyond the scope of the present paper.

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